

The **Discrepancy** of Shortest Paths



Greg Bodwin
U Michigan

Chengyuan Deng
Rutgers

Jie Gao
Rutgers

Gary Hoppenworth
U Michigan

Jalaj Upadhyay
Rutgers

Chen Wang
Rice

What is discrepancy?

Discrepancy $\leftrightarrow$ Shortest paths?

Why do we care?

What is discrepancy?

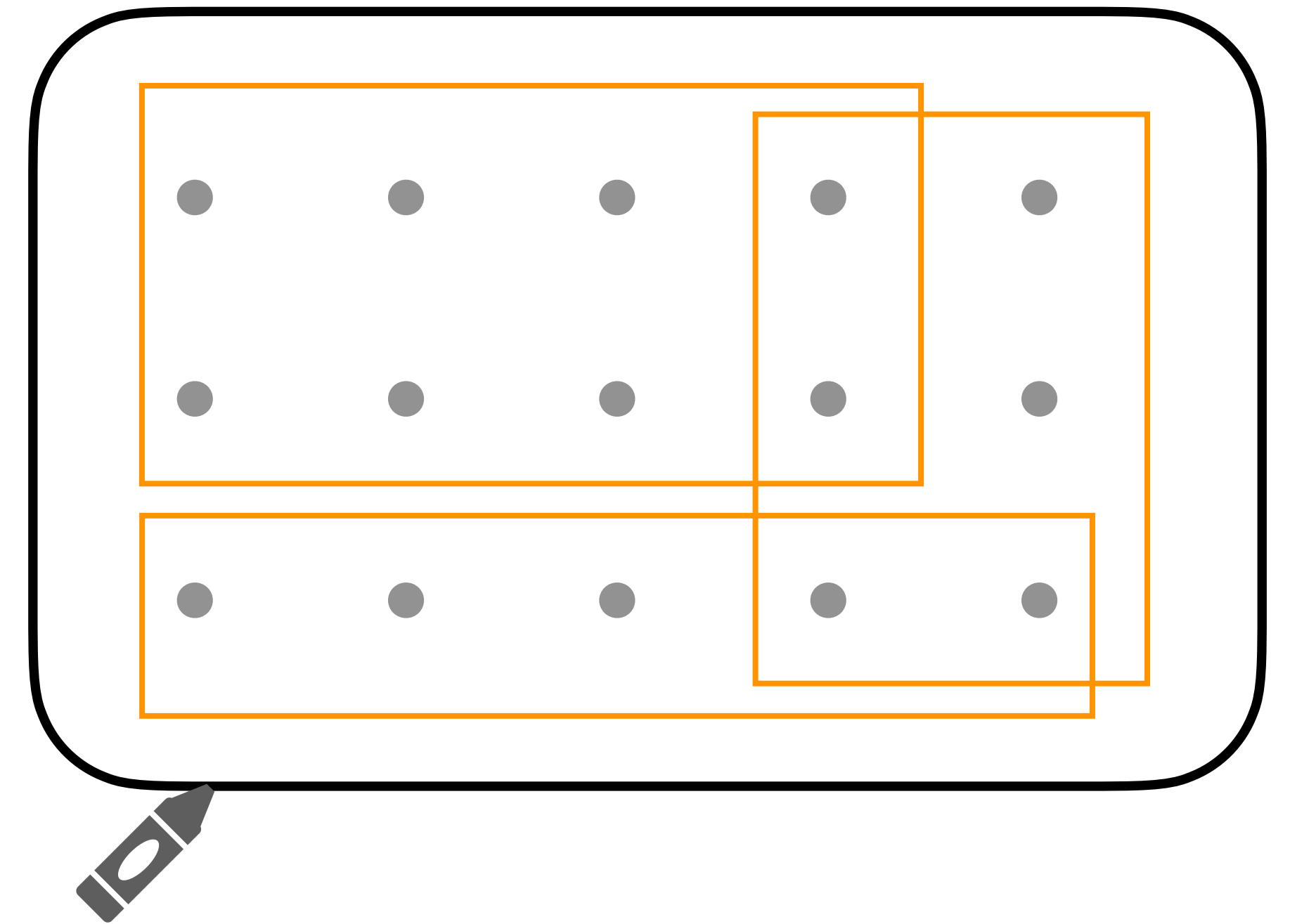


Discrepancy $\leftrightarrow$ Shortest paths?

Why do we care?

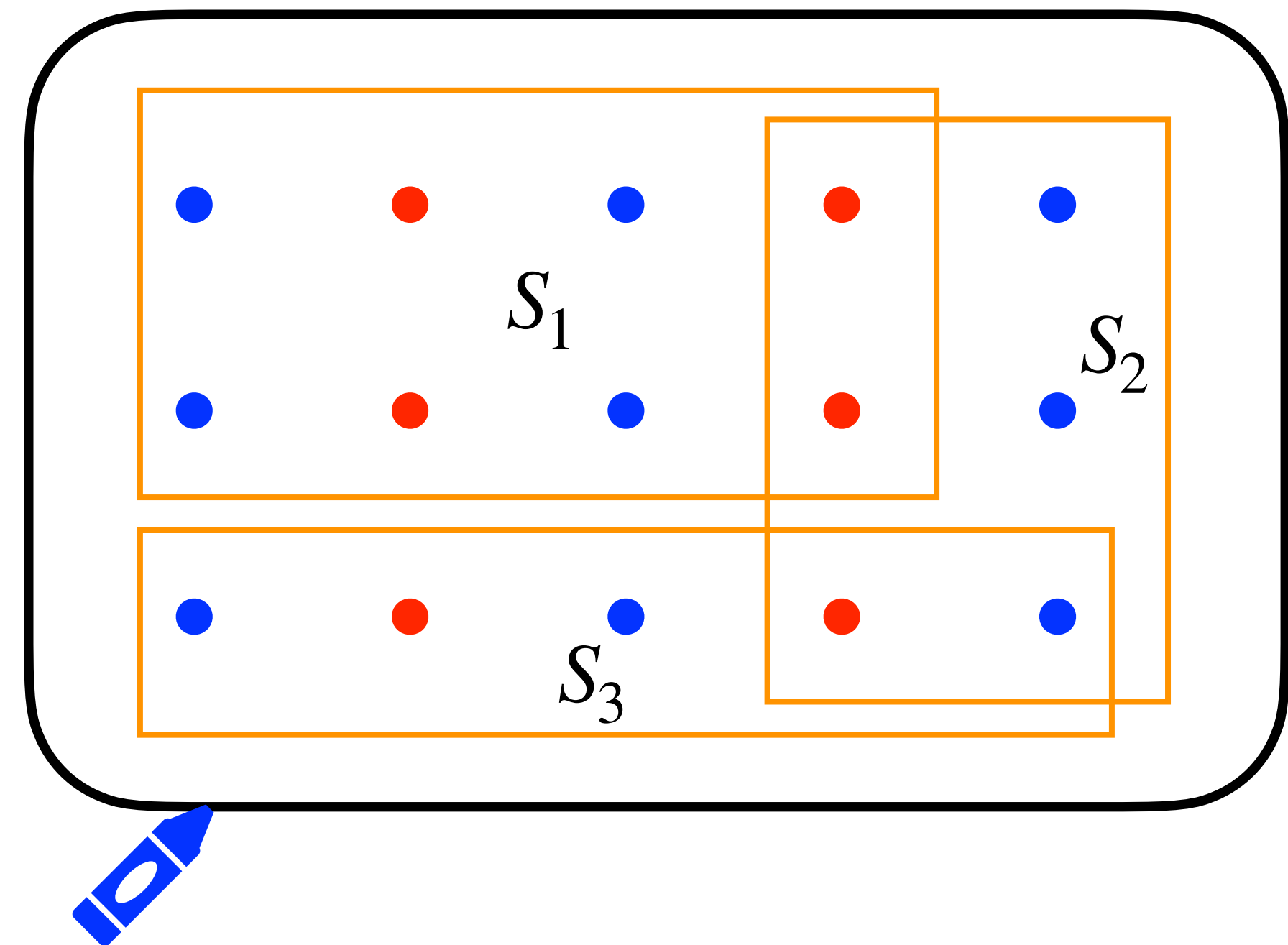
Discrepancy

- A set system $(X, \mathcal{S})$, $|X| = n$
- Two-color elements in X s.t.
in any $S \in \mathcal{S}$,
elements in different colors are roughly equal



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in any $S \in \mathcal{S}$,
elements in different colors are roughly equal



How do we quantify the “cost” of this coloring?

Count the difference!

Discrepancy

Definition:

- Set system $(X, \mathcal{S})$, coloring $\chi : X \rightarrow \{-1, +1\}$ for each element in X .

- For a set $S \in \mathcal{S}$, we denote $\chi(S) := \left| \sum_{i \in S} \chi(i) \right|$

The “cost” of one set S
w.r.t. a coloring χ

- The **discrepancy** of the set system is

$$\text{disc}(\mathcal{S}) := \min_{\chi: X \rightarrow \{-1, +1\}} \max_{S \in \mathcal{S}} |\chi(S)|$$

Find the best coloring s.t.
the worst cost is small

Fix the subsets first, then color

- Discrepancy is a measure of the “complexity” of the set system

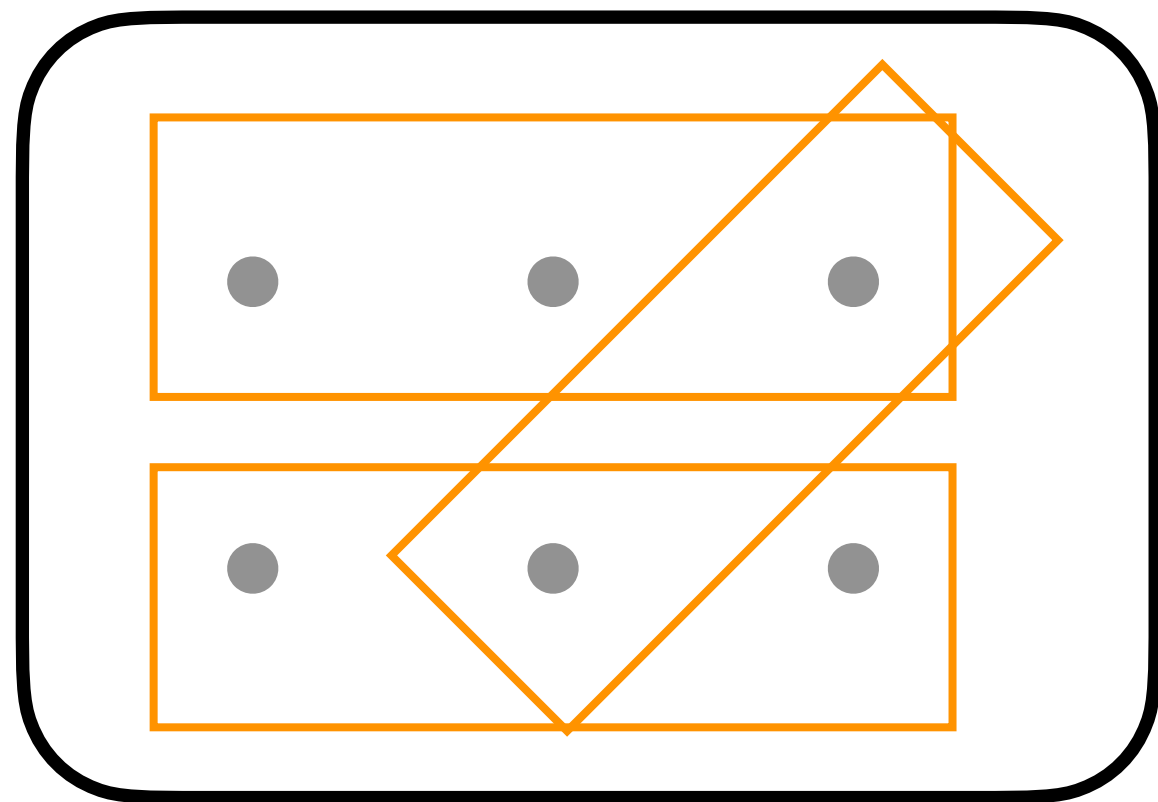
Discrepancy

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- Equivalently, let $A \in \mathbb{R}^{|\mathcal{S}| \times n}$ be the incidence matrix of the set system,

$$\text{disc}(A) = \text{disc}(\mathcal{S}) := \min_{x \in \{-1, +1\}^n} \|Ax\|_\infty$$



$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

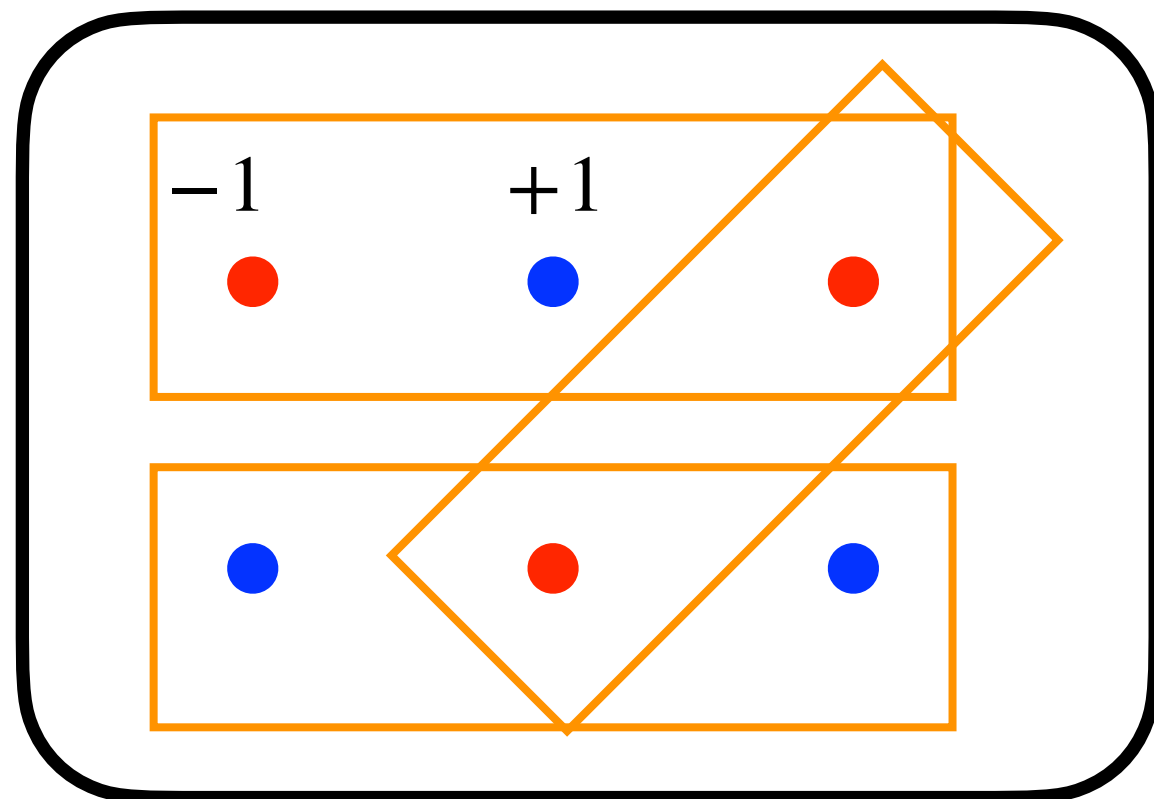
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$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$Ax = [1, -1, 2]$$

Hereditary Discrepancy

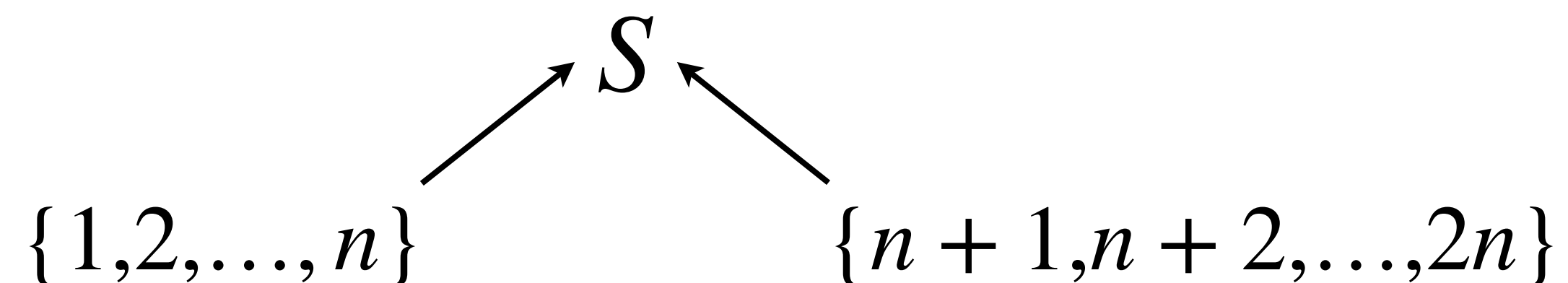
A more robust definition of discrepancy

Sometimes, you can get a lucky coloring

Consider the set system $(X, \mathcal{S})$:

$$X = \{1, 2, \dots, 2n\},$$

$$\mathcal{S} = \{S : |S \cap \{1, 2, \dots, n\}| = |S \cap \{n+1, \dots, 2n\}|\}$$



We can color X s.t. $\text{disc}(\mathcal{S}) = 0$

Hereditary Discrepancy

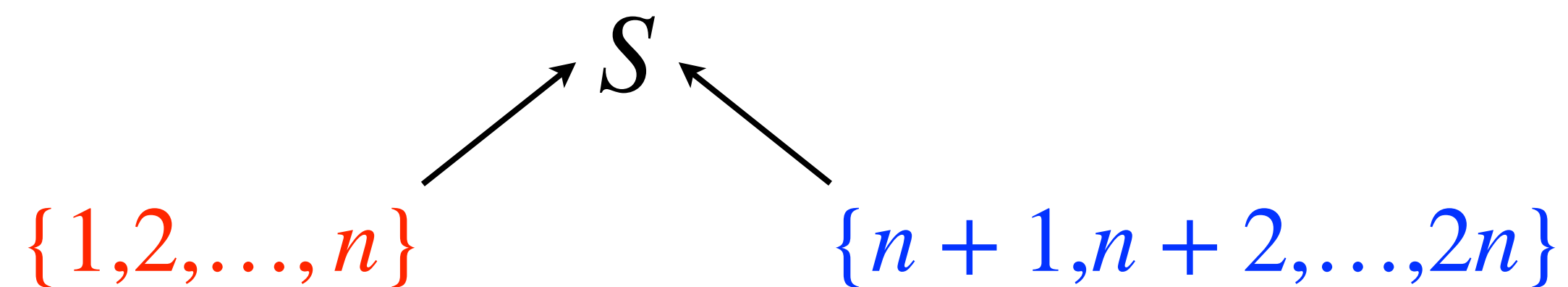
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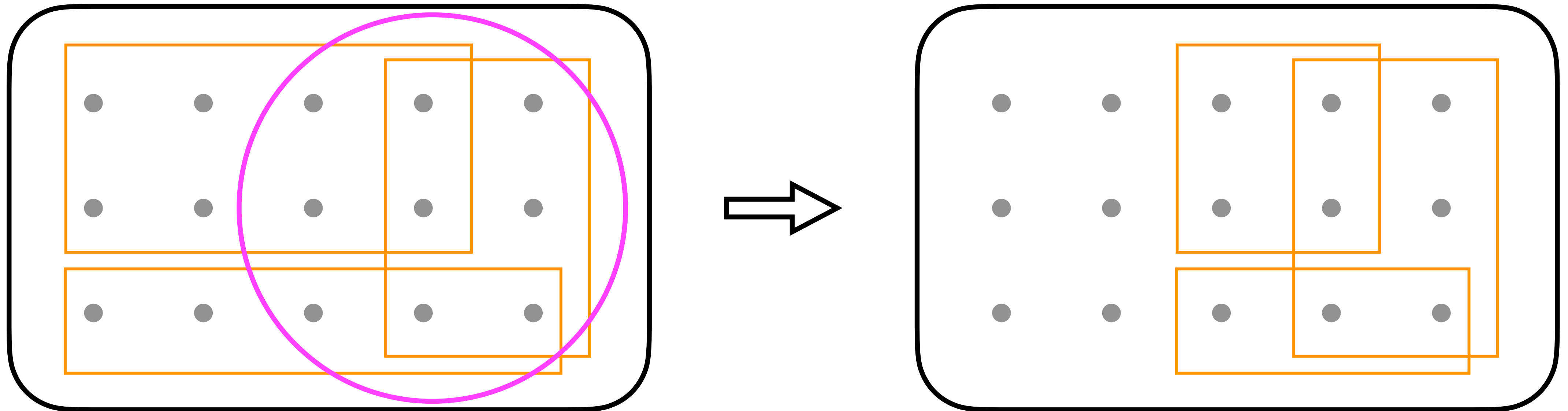
Hereditary Discrepancy

Definition: Given a set system $(X, \mathcal{S})$, its **hereditary discrepancy** is:

$$\text{herdisc}(\mathcal{S}) := \max_{Y \subseteq X} \text{disc}(\mathcal{S}_Y)$$

Clearly, $\text{herdisc}(\mathcal{S}) \geq \text{disc}(\mathcal{S})$

where $\mathcal{S}_Y$ is the sub-set system induced by Y



Hereditary Discrepancy

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where $\mathcal{S}_Y$ is the sub-set system induced by Y

Equivalently, let $A_Y \in \mathbb{R}^{|\mathcal{S}| \times |Y|}$ be the incidence matrix of the sub-set system,

$$\text{herdisc}(A) = \text{herdisc}(\mathcal{S}) := \max_{Y \subseteq X} \min_{x \in \{-1, +1\}^n} \|A_Y x\|_\infty$$

What is discrepancy?

A measure of complexity of a set system

Discrepancy $\leftrightarrow$ Shortest paths?



Why do we care?

Path System as Set System

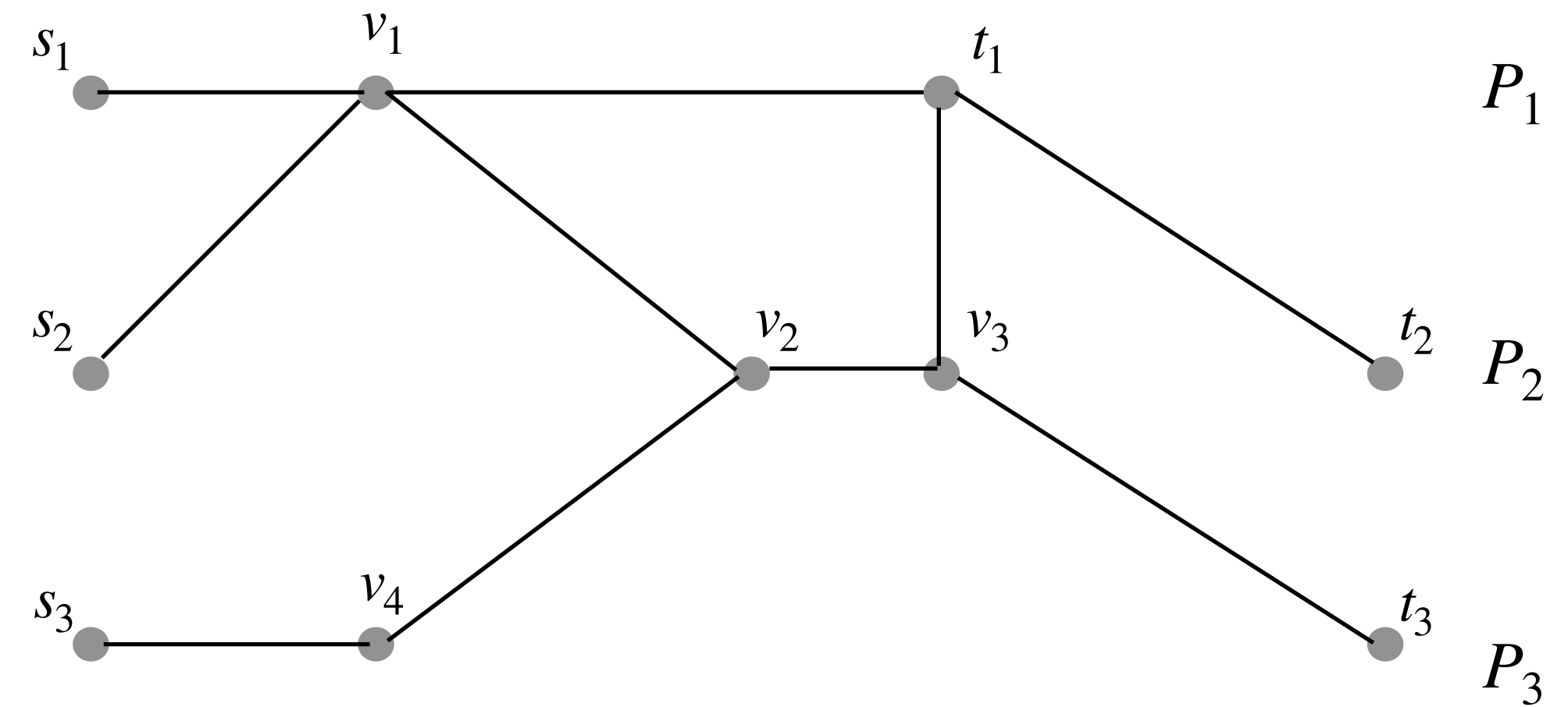
- A path system (V, Π) , $|V| = n$
- Consider each path $P \in \Pi$ as a set, whose elements are vertices in V

$$P_1 = \{s_1, v_1, t_1\}$$

$$P_2 = \{s_2, v_1, v_2, v_3, t_1, t_2\}$$

$$P_3 = \{s_3, v_4, v_2, v_3, t_3\}$$

- $(V, \Pi) \iff (X, \mathcal{S})$



Color the vertices!

Path System as Set System

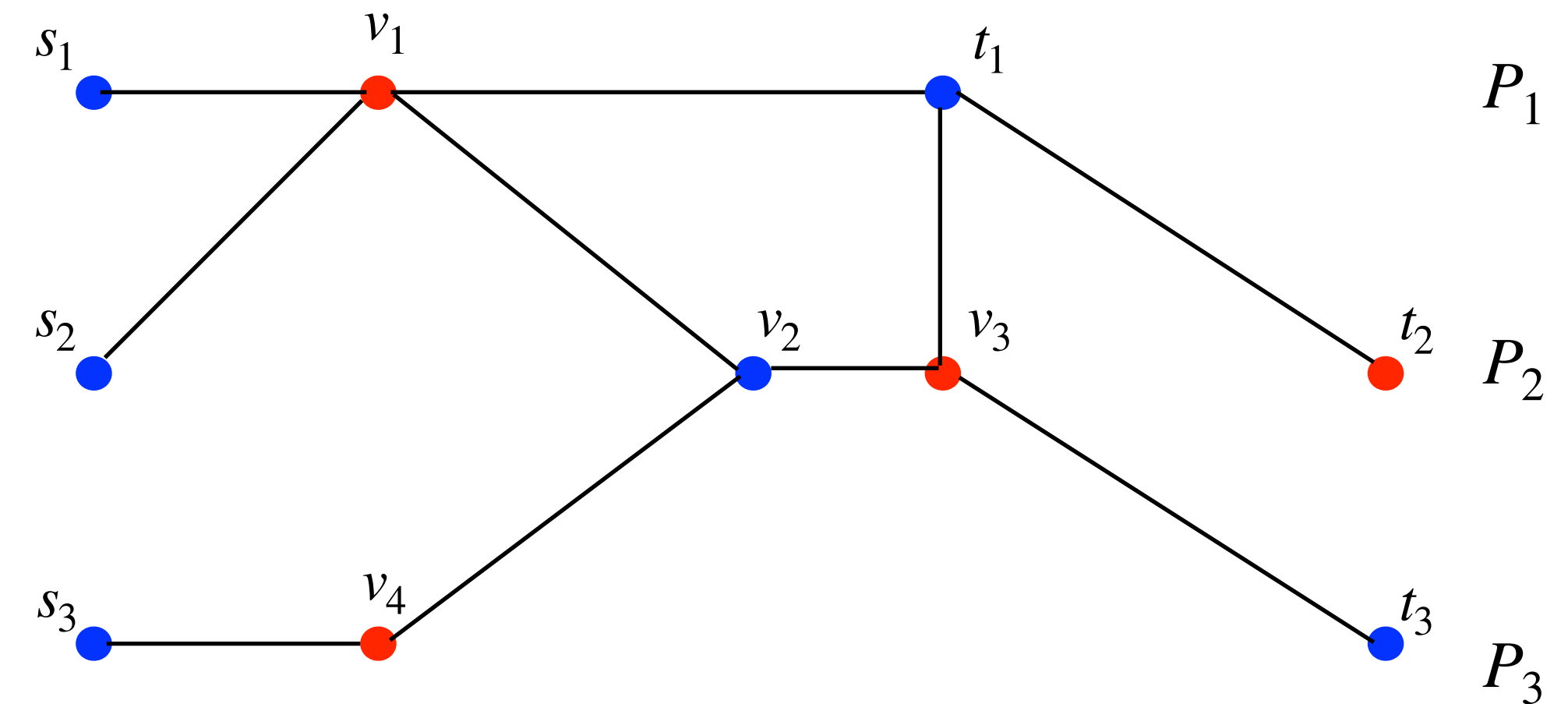
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- $(V, \Pi) \iff (X, \mathcal{S})$



$$\text{disc}_v(P_1) = \text{disc}_v(P_2) = 0$$

$$\text{disc}_v(P_3) = 1$$

$$\text{disc}_v(\mathcal{P}) = 1$$

Vertex discrepancy of $\mathcal{P}$ is 1

Similarly, we can consider edges as set elements

Path System as Set System

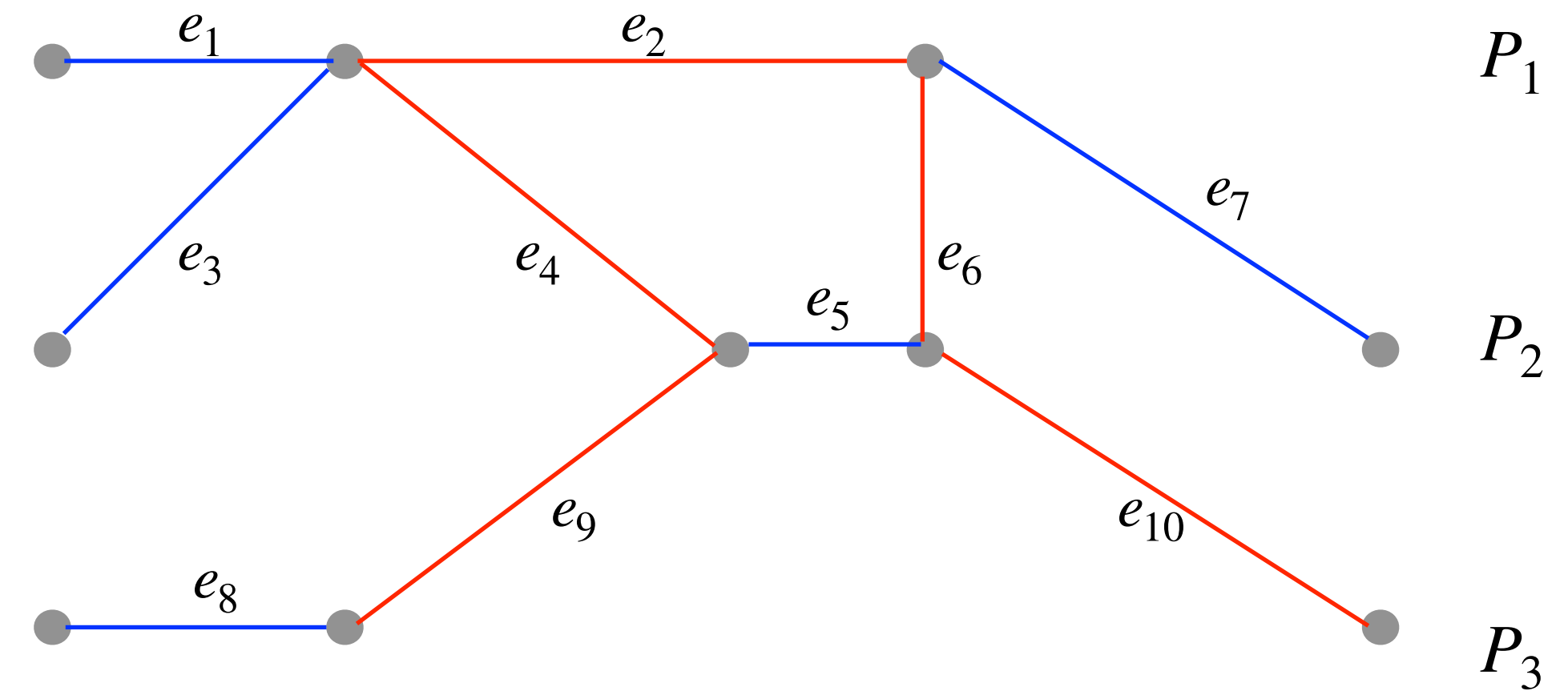
- A path system (E, Π) , $|E| = m$
- Consider each path $P \in \Pi$ as a set, whose elements are edges in E

$$P_1 = \{e_1, e_2\}$$

$$P_2 = \{e_3, e_4, e_5, e_6, e_7\}$$

$$P_3 = \{e_8, e_9, e_5, e_{10}\}$$

- $(E, \Pi) \iff (X, \mathcal{S})$



$$\text{disc}_e(P_1) = \text{disc}_e(P_3) = 0$$

$$\text{disc}_e(P_2) = 1$$

$$\text{disc}_e(\mathcal{P}) = 1$$

Edge discrepancy of $\mathcal{P}$ is 1

Vertex Discrepancy

Definition:

- Graph $G = (V, E)$, a set of paths Π in G
- Vertex incidence matrix $A \in \mathbb{R}^{|\Pi| \times |V|}$

For $v \in V$ and $\pi \in \Pi$, $A_{\pi,v} = 1$ if $v \in \pi$ and 0 o.w.

- Vertex discrepancy of Π is:

$$\text{disc}_v(A) = \text{disc}_v(\Pi) := \min_{x \in \{-1, +1\}^n} \|Ax\|_\infty$$

- Vertex hereditary discrepancy of Π is:

$$\text{herdisc}_v(A) = \text{herdisc}_v(\Pi) := \max_{Y \subseteq X} \text{disc}_v(A_Y)$$

Shortest Path Vertex Discrepancy

Definition:

- Graph $G = (V, E)$, a set of shortest paths Π in G
- Vertex incidence matrix $A \in \mathbb{R}^{|\Pi| \times |V|}$

For $v \in V$ and $\pi \in \Pi$, $A_{\pi,v} = 1$ if $v \in \pi$ and 0 o.w.

- Shortest path vertex discrepancy is:

$$\text{disc}_v(A) = \text{disc}_v(\Pi) := \min_{x \in \{-1, +1\}^n} \|Ax\|_\infty$$

- Shortest path vertex hereditary discrepancy is:

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What is discrepancy?

A measure of complexity of a set system

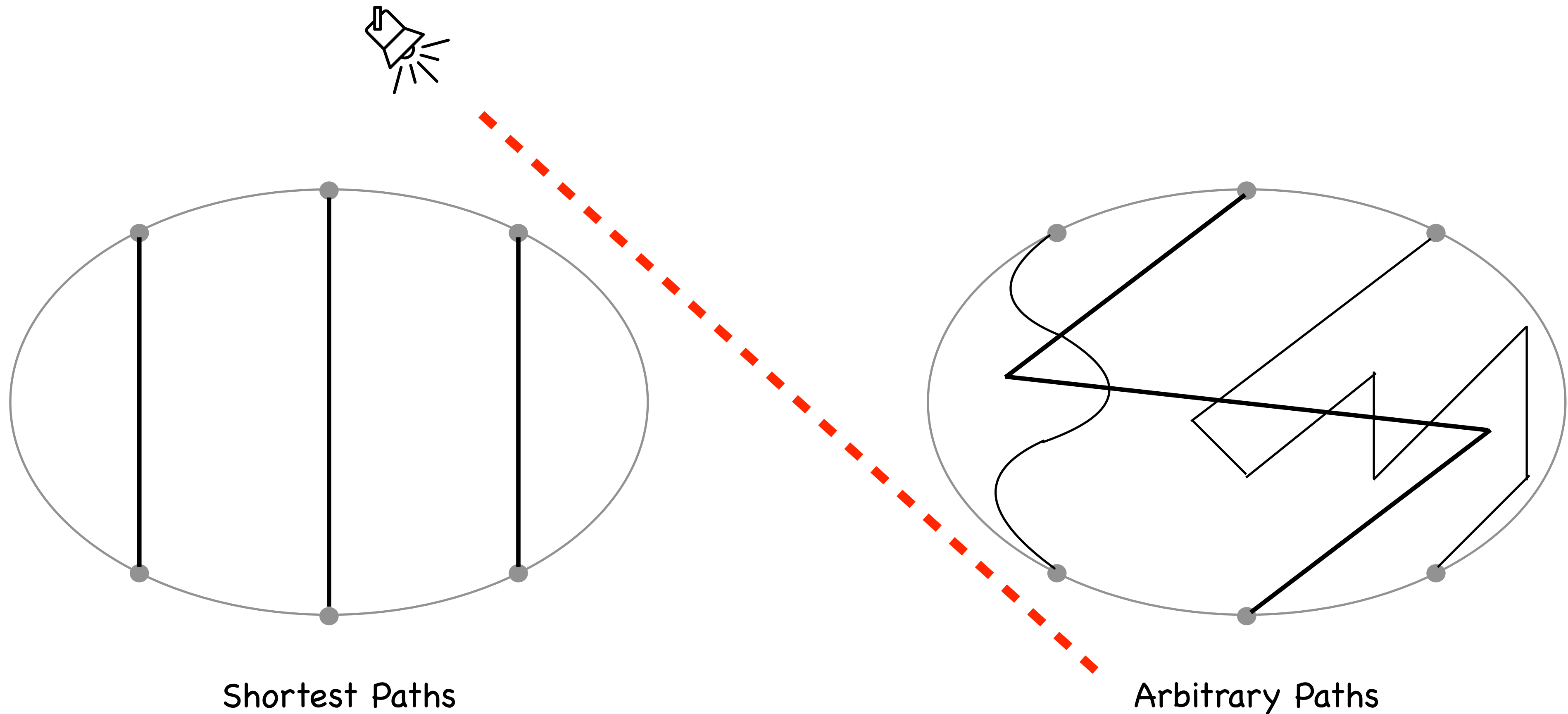
Discrepancy $\leftrightarrow$ Shortest paths?

Shortest path system as a set system
Assumption: shortest paths are unique

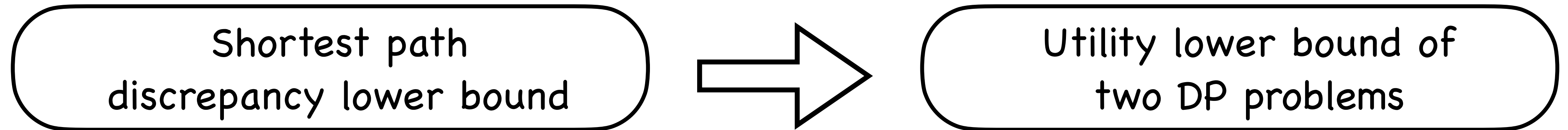
Why do we care?



Motivation 1: Any **structural separation** between shortest paths and arbitrary paths,
through the lens of discrepancy theory?



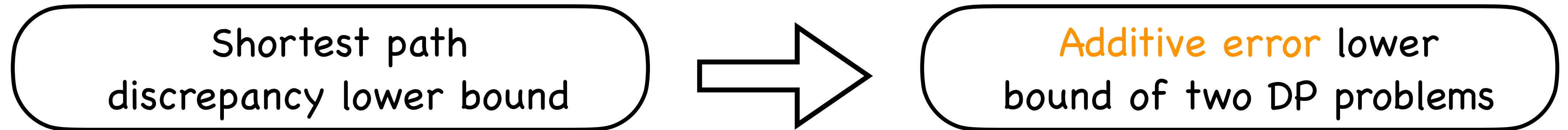
Motivation 2: Application to Differential Privacy



Differential privacy

- Given some data with sensitive information
- Release some computation results
- Goal:
 1. Protect sensitive information with DP
 2. Minimize the **error** on the output

Motivation 2: Application to Differential Privacy



Details later

Differential privacy

- Given some data with sensitive information
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 1. Protect sensitive information with DP
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Our Results

Main Theorem: For any undirected weighted graph G with a unique shortest path between each pair of nodes, let Π be the set of all shortest paths, then

$$\text{herdisc}_v(\Pi) = \text{herdisc}_e(\Pi) = \tilde{\Theta}(n^{1/4})$$

Remarks

- Tight up to polylogarithmic factors
- Our upper bound is constructive: we give a coloring algorithm
- Prior work: $\text{herdisc}(\Pi) \geq \Omega(n^{1/6})$ [CL00], based on a construction by Erdős
- Recall motivations
 1. **Structural separation: YES!** — More details on arbitrary paths
 2. Improve the lower bound of two DP problems to $\tilde{\Omega}(n^{1/4})$

Our Results: Structural Separation

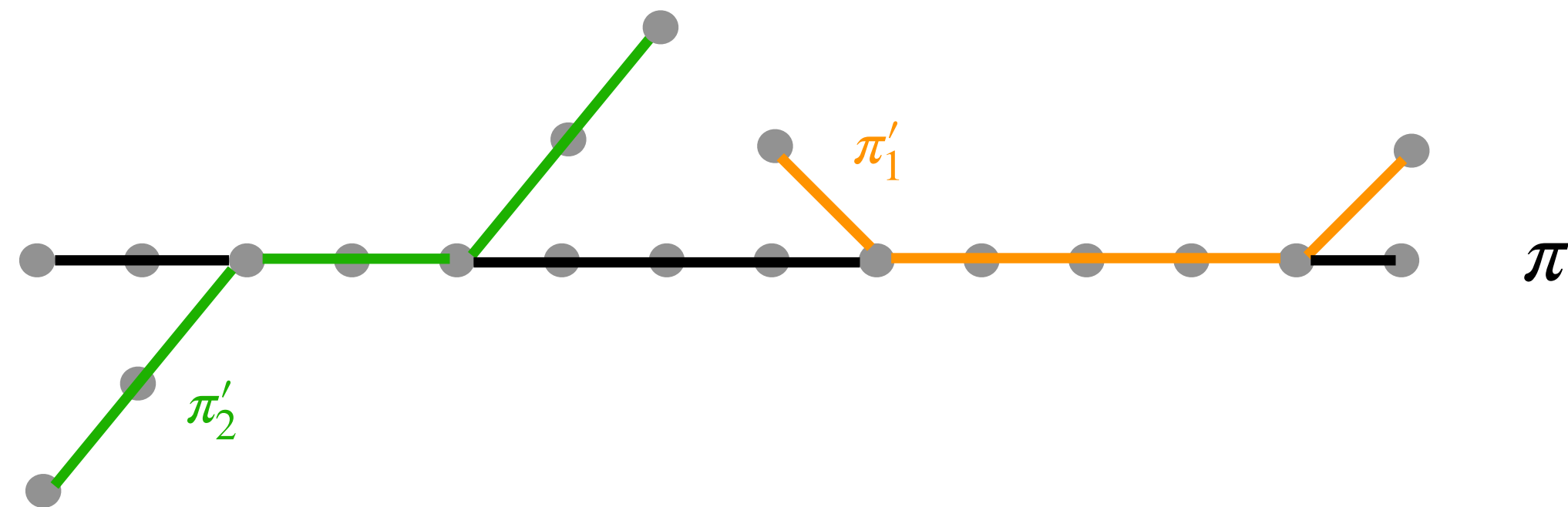
Contrasting the main result

- There are certain graphs s.t.
- A path system Π of **exponential** size can have $\text{disc}_v(\Pi) \geq \Omega(n)$
- A path system Π of **polynomial** size can have $\text{disc}_v(\Pi) \geq \Omega(\sqrt{n})$
- The shortest path system clearly has size of $O(n^2)$ Actually tight up to polylog factors
- Still, $\tilde{\Theta}(n^{1/4})$ **vs** $\tilde{\Theta}(n^{1/2})$

Our Results: Structural Separation

Is there a particular property of shortest paths leading to the separation?

- Yes! Shortest paths are **consistent**
- In fact, consistency is all we ask for



The main theorem applies to all consistent path systems!

Upper bound: vertex coloring



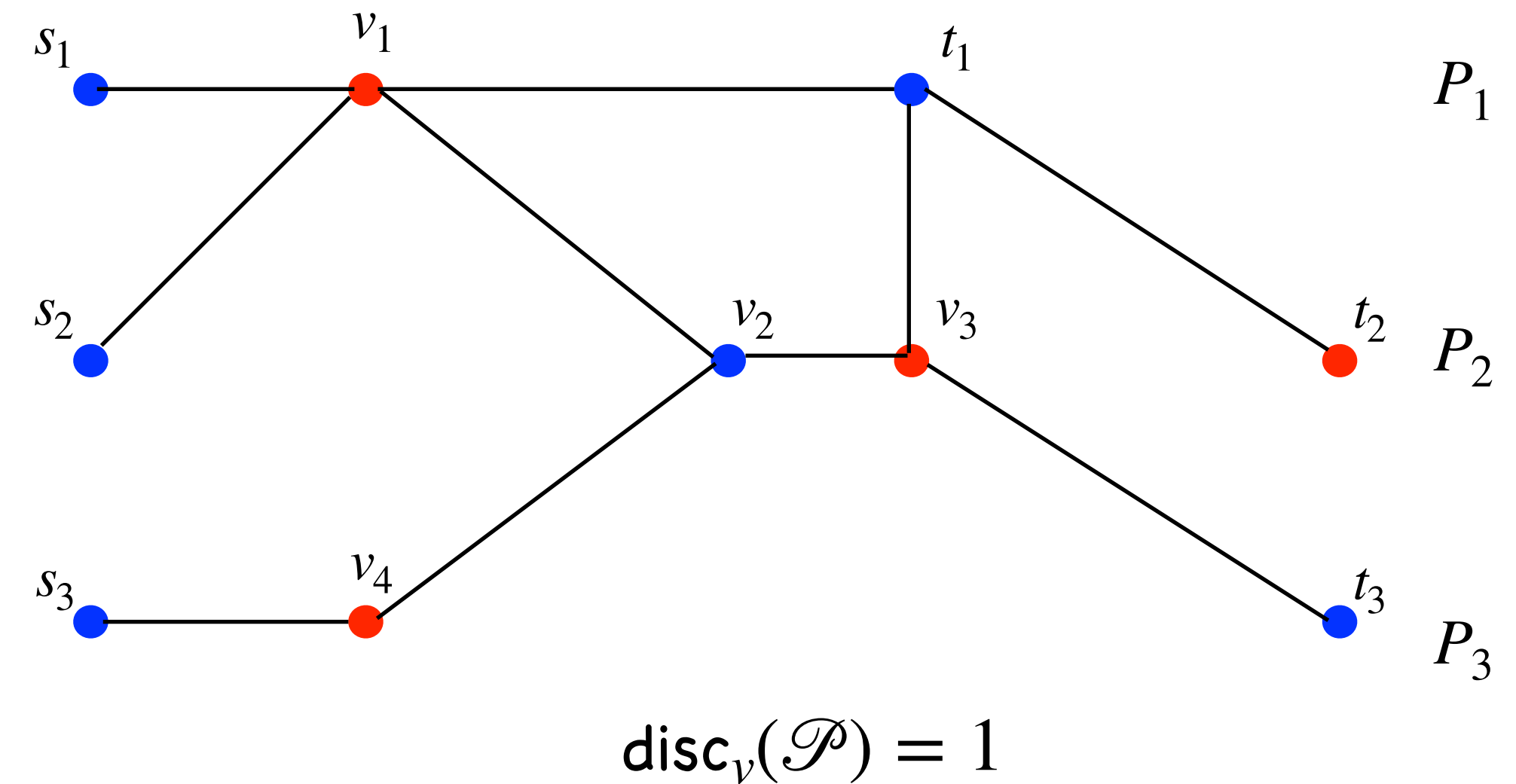
Lower bound: construction
and technique

Connection to Differential Privacy

Upper Bound: Vertex Coloring

Primitive schemes

- **Alternate coloring**
Color consecutive nodes with +1, -1 alternatively; the discrepancy is at most 1.
- We cannot do this too much



Upper Bound: Vertex Coloring

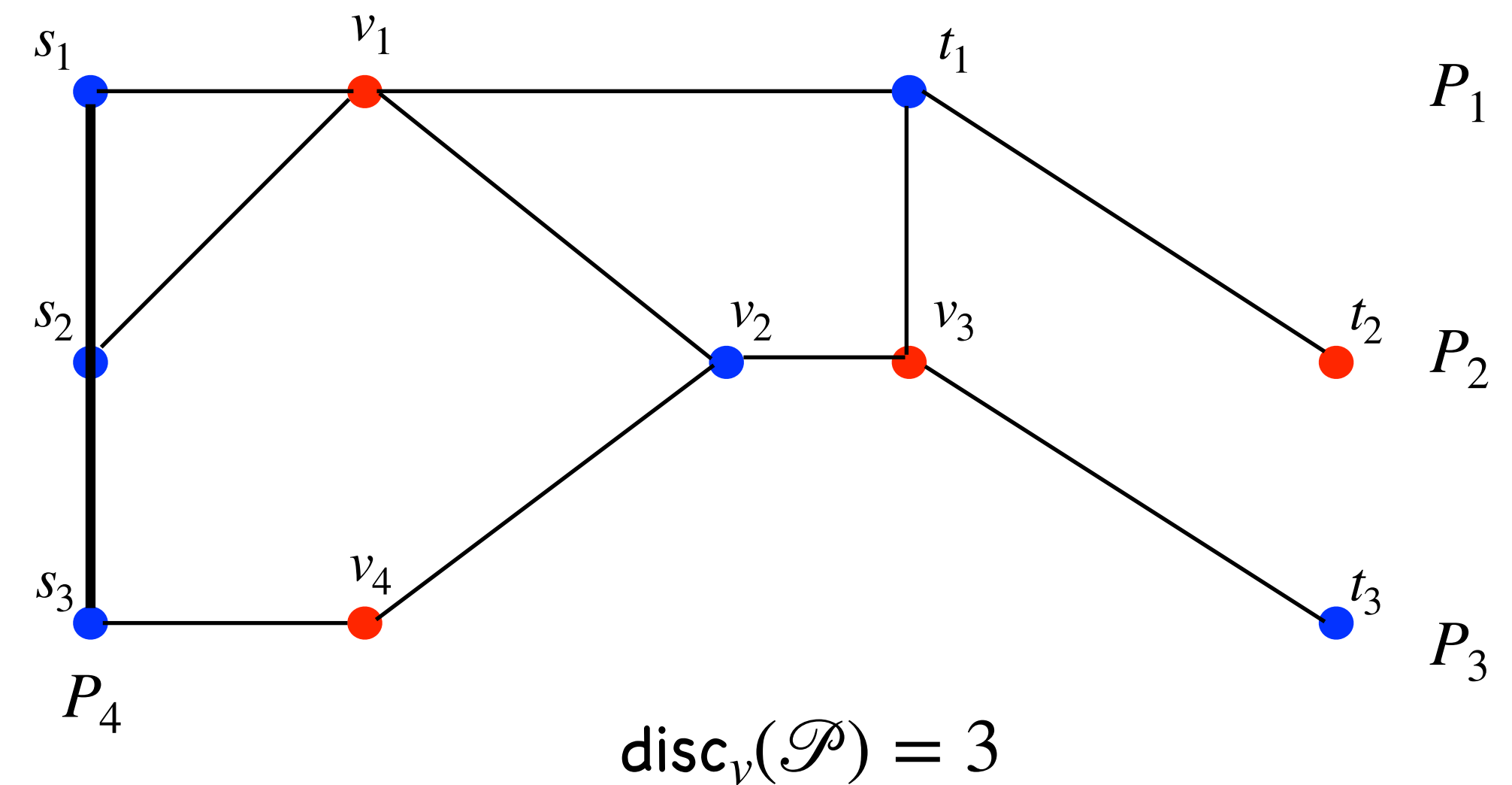
Primitive schemes

- **Alternate coloring**

Color consecutive nodes with +1, -1 alternatively; the discrepancy is at most 1.

- We cannot do this too much

- Guarantee a consistent coloring
- If a vertex is already colored, follow until no overlap
- Many overlaps $\rightarrow$ large discrepancy up to $O(n)$



Upper Bound: Vertex Coloring

Primitive schemes

- Random coloring [Cha00]

Using random coloring χ , then for any path $\pi \in \Pi$,

$$|\chi(\pi)| \leq \sqrt{2|\pi| \ln 4 |\Pi|}$$

Chernoff + Union Bound

- The shortest path vertex discrepancy can be $\tilde{O}(\sqrt{n})$ by random coloring

Upper Bound: Vertex Coloring

Can we use alternate coloring “as much as possible” without conflicts?

— Yes, but at most on $\sqrt{n}$ path intersections

The coloring is consistent



Our goal of construction

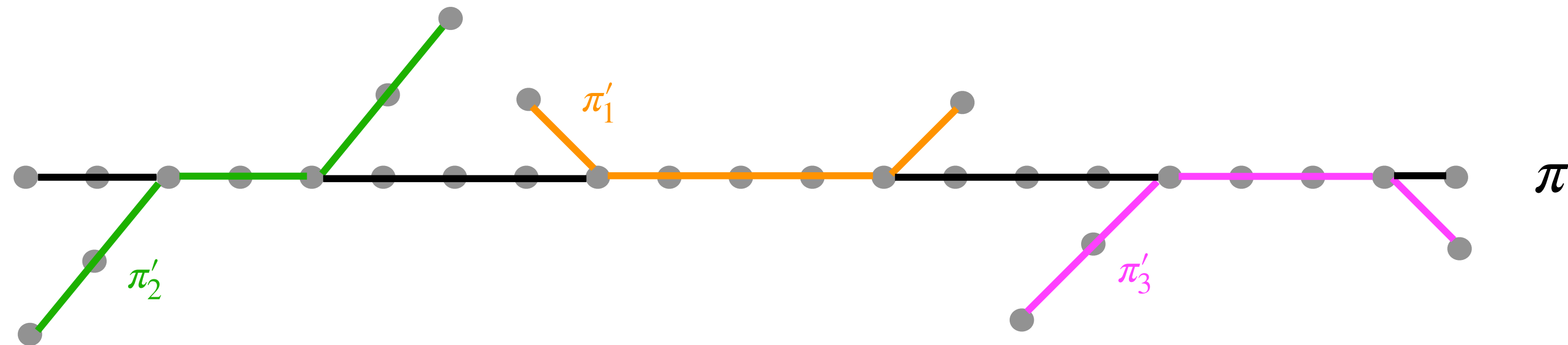
- Any shortest path π , other paths π'_i intersect with π , **vertex disjoint**

Upper Bound: Vertex Coloring

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Our goal of construction

- Any shortest path π , other paths π'_i intersect with π , **vertex disjoint**

- Not too many intersections

- Not too many rest vertices on π

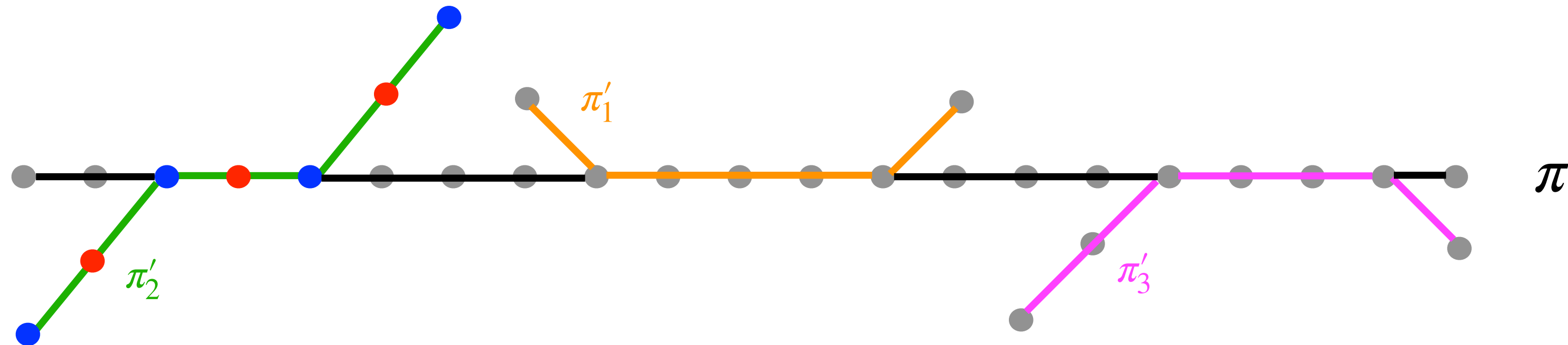
Shatter all shortest paths

Upper Bound: Vertex Coloring

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Main Idea

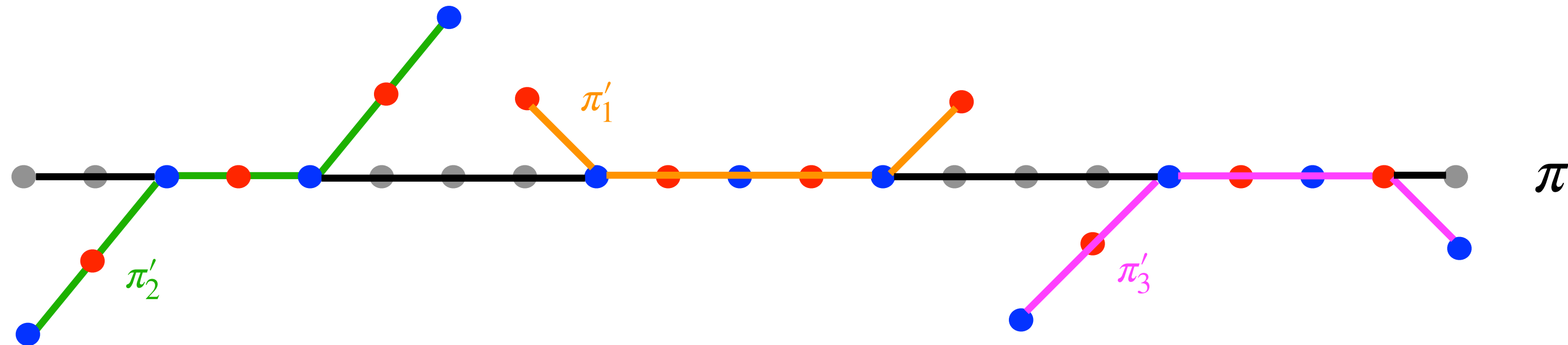
- For each π'_i apply **alternate coloring**, start randomly, $\text{disc}_v(\pi'_i) \leq 1$

Upper Bound: Vertex Coloring

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Main Idea

- For each π'_i apply **alternate coloring**, **start randomly**, $\text{disc}_v(\pi'_i) \leq 1$
- If there are k path intersections, the contribution to π is at most $\sqrt{k}$
- For the rest vertices, **random coloring** is (hopefully) affordable

Upper Bound: Vertex Coloring

Formally, we apply the path cover technique [KP22]

“Greedy” approach:

- For any path π with $\geq \sqrt{n}$ unshattered vertices
- Pick (sub)path π' of size $\sqrt{n}$, shatter it
- π' paths are vertex-disjoint

Properties of Π' (all π' paths)

1. $|\Pi'| \leq \sqrt{n}$
2. Any path π has at most $\sqrt{n}$ vertices not shattered
3. Paths in Π' are pair-wise vertex disjoint
4. Path intersections are contiguous subpaths of π'

Decide the exponent on n

Ensure a consistent coloring

Upper Bound: Vertex Coloring

Bounding the discrepancy of two sources

1. Discrepancy of **alternate coloring** (**shattered vertices on π**)

$$|\Pi'| \leq \sqrt{n} \text{ (Property 1) gives } O(n^{1/4} \log^{1/2} n)$$

2. Discrepancy of **random coloring** (**unshattered vertices on π**)

$$|\pi \setminus V'| \leq \sqrt{n} \text{ (Property 2) also gives } O(n^{1/4} \log^{1/2} n)$$

Recall: The discrepancy upper bound applies to all consistent path systems

Extending to hereditary discrepancy:

The induced sub-path system is also consistent

Upper bound: vertex coloring

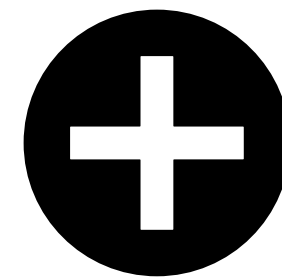
Lower bound: construction
and technique

Connection to Differential Privacy

Lower Bound

Construction

Point-line system
by Paul Erdős



Technique

Trace Bound [CL00,
Lar19] for hereditary
discrepancy

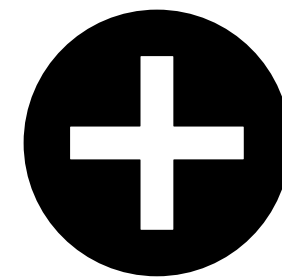
This gives the $\Omega(n^{1/6})$ lower bound

Lower Bound

Construction

Lower bound instance
by [Greg Bodwin and
Gary Hoppenworth23]

Exact hopset lower bound



Technique

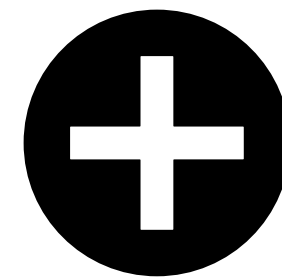
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This gives the $\tilde{\Omega}(n^{1/4})$ lower bound

Lower Bound

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Technique

Trace Bound [CL00,
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discrepancy

Proposition 1 (Trace Bound): If A is an $m \times n$ incidence matrix and $M = A^T A$, it holds that:

$$\text{herdisc}(A) \geq \frac{(\text{tr}(M))^2}{8e \text{tr}(M^2) \cdot \min\{m, n\}} \sqrt{\frac{\text{tr}(M)}{\max\{m, n\}}}$$

Think about $m > n$

Lower Bound: Point-line system

Properties of Construction by Erdős

- n points and n lines in the plane
 1. Each point lies on $\Theta(n^{1/3})$ lines
 2. Each line passes $\Theta(n^{1/3})$ points

To apply the trace bound, we need to compute $\text{tr}(M)$, $\text{tr}(M^2)$

- $\text{tr}(M) = n \cdot \Theta(n^{1/3}) = \Theta(n^{4/3})$

- $\text{tr}(M^2) = \text{\# 4-cycles} = \sum_{j=1}^n \sum_{\substack{u, v \in P_j \\ u \neq v}} m_{u,v} + \sum_{i=1}^n m_{ii}^2$ Degenerated cycles

$$= \text{\# pairs of vertices on } n \text{ lines} + n \cdot n^{2/3} = O(n^{5/3})$$

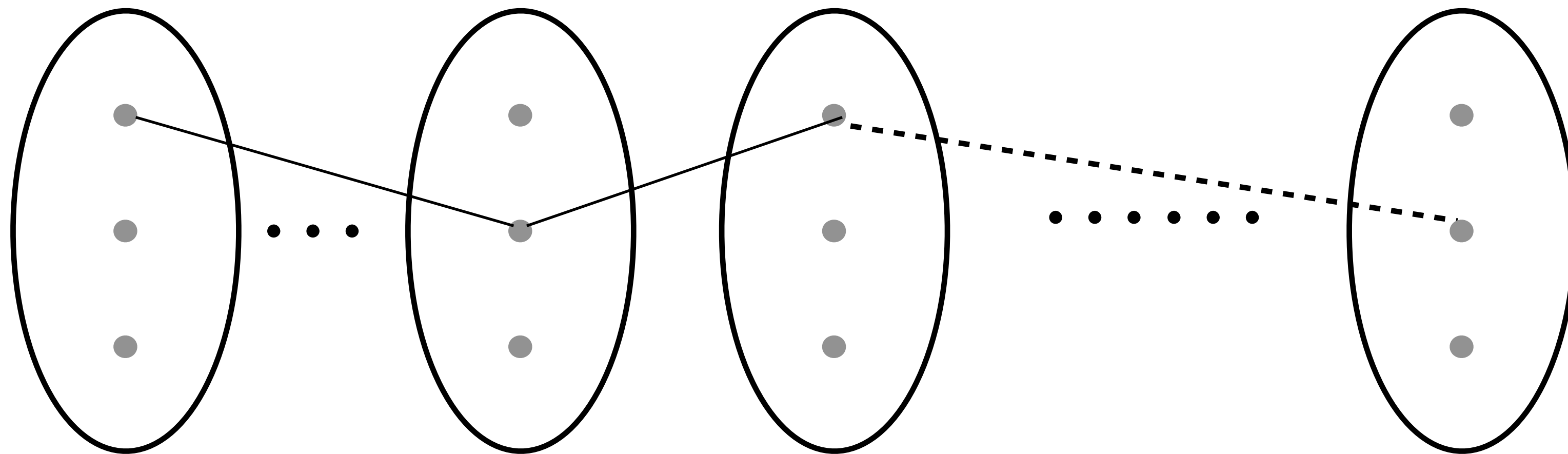
- The trace bound gives $\Omega(n^{1/6})$

Lower Bound: Construction

Properties of Construction by [BH23]

- For any $p \in [1, n^2]$, we can construct undirected graph $G = (V, E, w)$ and a set Π of p paths in G s.t.
 1. G is a layered graph, with $\ell = \Theta(n/\sqrt{p \log n})$ layers.

Each path in Π starts in the first layer, ends in the last layer, and contains exactly one node in each layer.

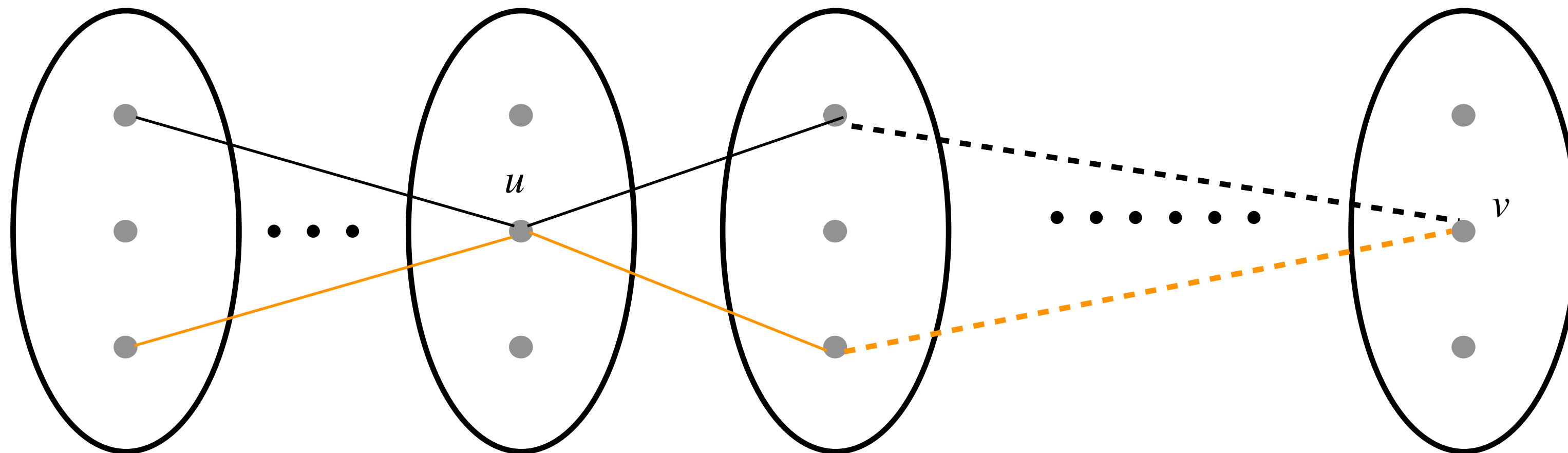


Lower Bound: Construction

Properties of Construction by [BH23]

- For any $p \in [1, n^2]$, we can construct undirected graph $G = (V, E, w)$ and a set Π of p paths in G s.t.
 1. G is a layered graph, with $\ell = \Theta(n/\sqrt{p \log n})$ layers.
 2. Each path in Π is the **unique shortest path** between its endpoints in G
 3. For $u, v \in V$, there are at most $\frac{\ell}{h(u, v)}$ paths in Π that contain both u, v

Hop distance between u, v



Lower Bound: Computation

Let A of size $p \times n$ be the incidence matrix of vertices in Π , we just need to compute $\text{tr}(M)$, $\text{tr}(M^2)$

- $\text{tr}(M) = p\ell$ (#1 in matrix A)

- $\text{tr}(M^2) = \sum_{j=1}^p \sum_{\substack{u, v \in P_j, \\ u \neq v}} m_{u,v} + \sum_{i=1}^n m_{ii}^2 = \sum_{j=1}^p \sum_{i=1}^{\ell} \sum_{\substack{u, v \in P_j, \\ h(u,v) = i}} m_{u,v} + n \cdot \left(\frac{p\ell}{n} \right)^2$

- Above $\leq \sum_{j=1}^p \sum_{i=1}^{\ell} \ell \cdot \frac{\ell}{i} + \frac{p^2 \ell^2}{n} \leq p\ell^2 \log \ell + \frac{p^2 \ell^2}{n}$

Property 3: For $u, v \in V$, there are at most $\frac{\ell}{h(u,v)}$ paths in Π that contain both u, v

Lower Bound: Computation

We have $\text{tr}(M) = p\ell$, $\text{tr}(M^2) \leq p\ell^2 \log \ell + \frac{p^2 \ell^2}{n}$

Recall $m = p$, we set $p = n \log n$, then $\ell = \Theta(n/\sqrt{p \log n}) = \Theta(\sqrt{n^{1/2}/\log n})$

Apply the trace bound:

$$\text{herdisc}_v(A) \geq \frac{(\text{tr}(M))^2}{8e \text{tr}(M^2) \cdot n} \sqrt{\frac{\text{tr}(M)}{p}} \geq \Omega(\sqrt{\ell}) = \Omega\left(\frac{n^{1/4}}{\sqrt{\log n}}\right)$$

Proposition 1 (Trace Bound): If A is an $m \times n$ incidence matrix and $M = A^T A$, it holds that:

$$\text{herdisc}(A) \geq \frac{(\text{tr}(M))^2}{8e \text{tr}(M^2) \cdot \min\{m, n\}} \sqrt{\frac{\text{tr}(M)}{\max\{m, n\}}}$$

Upper bound: vertex coloring

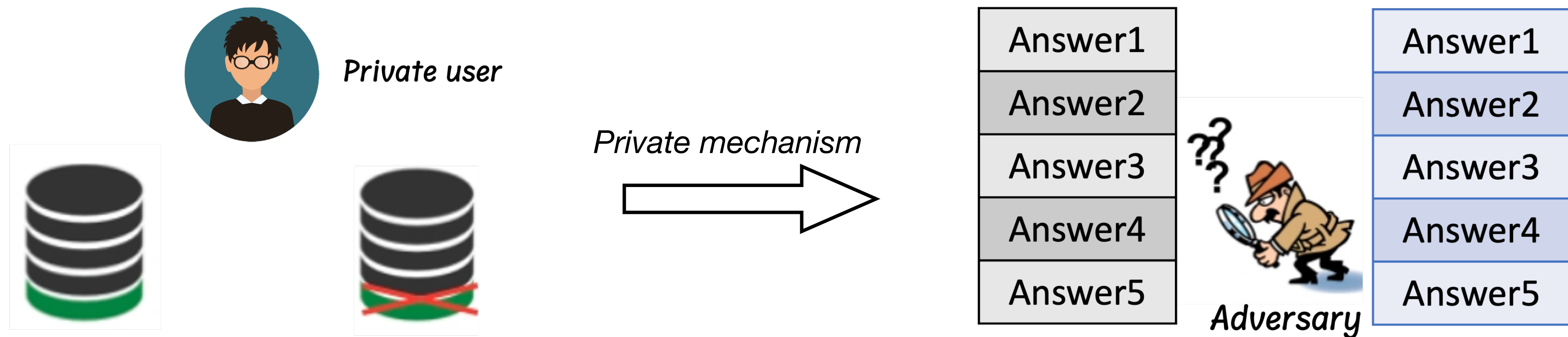
Lower bound: construction
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Connection to Differential Privacy



Differential Privacy

Key idea: Perturb the data s.t. for **neighboring datasets** differed by one instance, the adversary cannot distinguish the outcome



Compromise: error in the reported outcome

DP-All Pairs Shortest Distances

Given a weighted graph $G = (V, E, w)$, we can have matrix $D \in \mathbb{R}^{n \times n}$, where $D_{u,v}$ is the shortest distance between u, v

Goal of DP-APSD:

Release D' s.t.

1. The error is small, i.e. minimize $\|D - D'\|_\infty$
2. For **weight-neighboring** graphs, DP is guaranteed

ℓ_1 difference is at most one

The 2nd problem is DP All Sets Range Query, a weaker version of APSD

DP-All Pairs Shortest Distances

Definition:

- Graph $G = (V, E)$, any neighboring weights $w \sim w'$ and all sets of possible output $\mathcal{C}$
- A randomized algorithm $\mathcal{A}$ is (ϵ, δ) -differentially private on G if:

$$\Pr[\mathcal{A}(G, w) \in \mathcal{C}] \leq e^\epsilon \cdot \Pr[\mathcal{A}(G, w') \in \mathcal{C}] + \delta$$

Known results

1. Upper Bounds [CGK+23, FL23]

$\tilde{O}(n^{2/3})$ additive error for $(\epsilon, 0)$ -DP

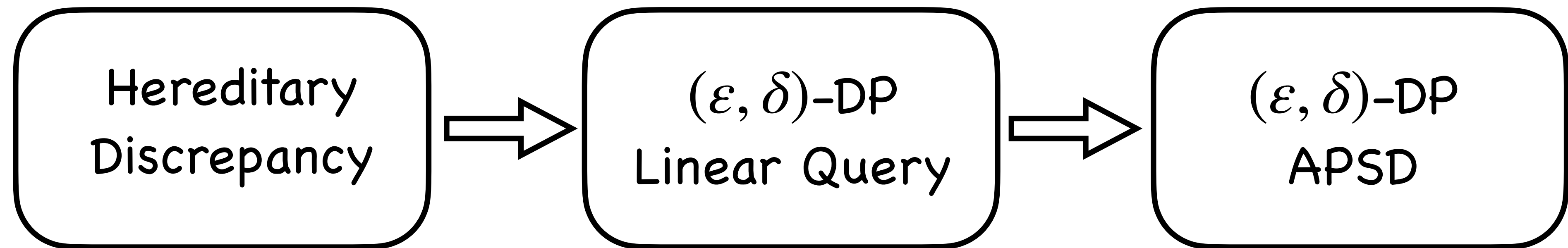
$\tilde{O}(n^{1/2})$ additive error for (ϵ, δ) -DP

2. Lower bound [CGK+23]

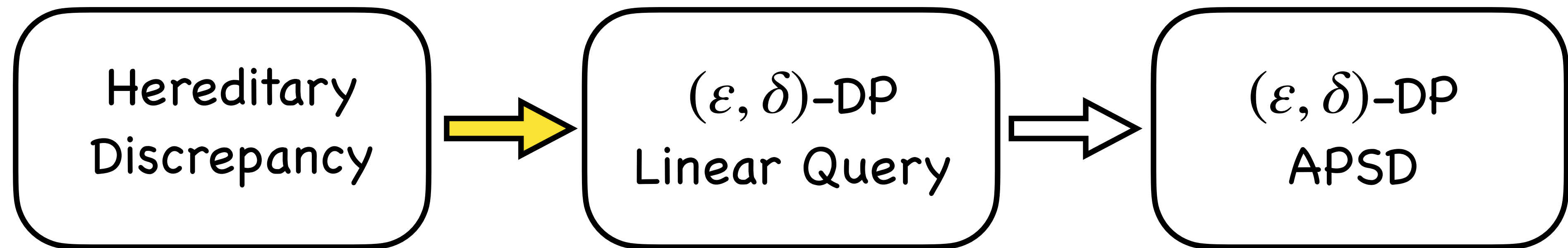
$\Omega(n^{1/6})$ additive error for (ϵ, δ) -DP

Our new discrepancy lower bound
directly improves this to $\tilde{\Omega}(n^{1/4})$

DP-APSD: Reduction

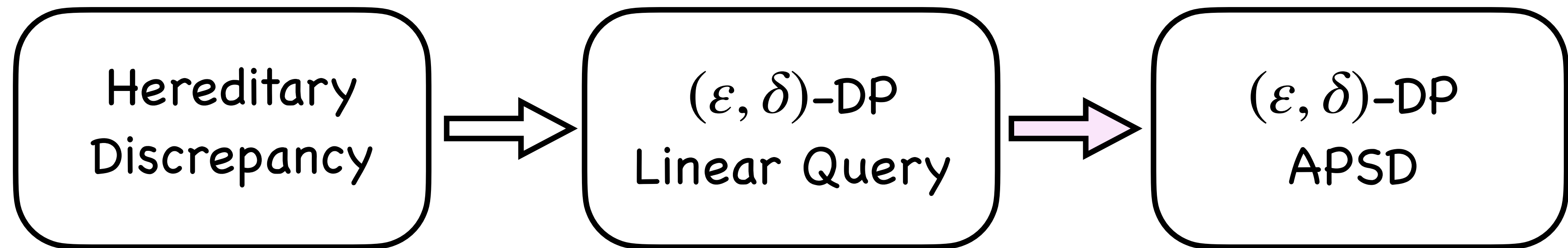


DP-APSD: Reduction



[MN12] Any (ϵ, δ) -DP algorithm for the $\mathbf{A}$ -query problem must have additive error at least $\text{herdisc}(\mathbf{A})/2$

DP-APSD: Reduction



[CGK+23] An (ε, δ) -DP algorithm for the **APSD** problem on a graph of size $2n$ with additive error α , implies an (ε, δ) -DP algorithm for the **A-linear query** problem on a graph of size n with additive error α .

Our Results

		Tree	Bipartite	Planar Graph	Undirected Graph	Directed Graph
Vertex	Discrepancy	$\Theta(1)$	$\Theta(1)$	$O(n^{1/4})$	$\Theta(n^{1/4})$	$\Theta(n^{1/4})$
	Hereditary Disc	$\Theta(1)$	$\tilde{\Theta}(n^{1/4})$	$\tilde{\Theta}(n^{1/4})$	$\Omega(n^{1/6})[\text{CL00}] \rightarrow \tilde{\Theta}(n^{1/4})$	$\tilde{\Omega}(n^{1/4})$
Edge	Discrepancy	$\Theta(1)$	$\Theta(1)$	$O(n^{1/4})$	$O(n^{1/4})$	$\min \{O(m^{1/4}), O(D^{1/2})\}$
	Hereditary Disc	$\Theta(1)$	$\tilde{\Theta}(n^{1/4})$	$\tilde{\Theta}(n^{1/4})$	$\Omega(n^{1/6})[\text{CL00}] \rightarrow \tilde{\Theta}(n^{1/4})$	$\tilde{\Omega}(n^{1/4})$

Main Result

How about other family of undirected graphs?

Our Results

		Tree	Bipartite	Planar Graph	Undirected Graph	Directed Graph
Vertex	Discrepancy	$\Theta(1)$	$\Theta(1)$	$O(n^{1/4})$	$\Theta(n^{1/4})$	$\Theta(n^{1/4})$
	Hereditary Disc	$\Theta(1)$	$\tilde{\Theta}(n^{1/4})$	$\tilde{\Theta}(n^{1/4})$	$\Omega(n^{1/6})[\text{CL00}] \rightarrow \tilde{\Theta}(n^{1/4})$	$\tilde{\Omega}(n^{1/4})$
Edge	Discrepancy	$\Theta(1)$	$\Theta(1)$	$O(n^{1/4})$	$O(n^{1/4})$	$\min \{O(m^{1/4}), O(D^{1/2})\}$
	Hereditary Disc	$\Theta(1)$	$\tilde{\Theta}(n^{1/4})$	$\tilde{\Theta}(n^{1/4})$	$\Omega(n^{1/6})[\text{CL00}] \rightarrow \tilde{\Theta}(n^{1/4})$	$\tilde{\Omega}(n^{1/4})$

Easy, alternate coloring from the root

Our Results

		Tree	Bipartite	Planar Graph	Undirected Graph	Directed Graph
Vertex	Discrepancy	$\Theta(1)$	$\Theta(1)$	$O(n^{1/4})$	$\Theta(n^{1/4})$	$\Theta(n^{1/4})$
	Hereditary Disc	$\Theta(1)$	$\tilde{\Theta}(n^{1/4})$	$\tilde{\Theta}(n^{1/4})$	$\Omega(n^{1/6})[\text{CL00}] \rightarrow \tilde{\Theta}(n^{1/4})$	$\tilde{\Omega}(n^{1/4})$
Edge	Discrepancy	$\Theta(1)$	$\Theta(1)$	$O(n^{1/4})$	$O(n^{1/4})$	$\min \{O(m^{1/4}), O(D^{1/2})\}$
	Hereditary Disc	$\Theta(1)$	$\tilde{\Theta}(n^{1/4})$	$\tilde{\Theta}(n^{1/4})$	$\Omega(n^{1/6})[\text{CL00}] \rightarrow \tilde{\Theta}(n^{1/4})$	$\tilde{\Omega}(n^{1/4})$

The herdisc bounds still hold!
Some non-trivial technique required

Our Results

		Tree	Bipartite	Planar Graph	Undirected Graph	Directed Graph
Vertex	Discrepancy	$\Theta(1)$	$\Theta(1)$	$O(n^{1/4})$	$\Theta(n^{1/4})$	$\Theta(n^{1/4})$
	Hereditary Disc	$\Theta(1)$	$\tilde{\Theta}(n^{1/4})$	$\tilde{\Theta}(n^{1/4})$	$\Omega(n^{1/6})[\text{CL00}] \rightarrow \tilde{\Theta}(n^{1/4})$	$\tilde{\Omega}(n^{1/4})$
Edge	Discrepancy	$\Theta(1)$	$\Theta(1)$	$O(n^{1/4})$	$O(n^{1/4})$	$\min \{O(m^{1/4}), O(D^{1/2})\}$
	Hereditary Disc	$\Theta(1)$	$\tilde{\Theta}(n^{1/4})$	$\tilde{\Theta}(n^{1/4})$	$\Omega(n^{1/6})[\text{CL00}] \rightarrow \tilde{\Theta}(n^{1/4})$	$\tilde{\Omega}(n^{1/4})$

All undirected

Open Problem

Upper bounds are based on VC-dimension technique, it is only an existential upper bound

Summary

- We show a structural separation between shortest paths and arbitrary paths, through the lens of discrepancy:

$$\text{herdisc}(\Pi) = \tilde{\Theta}(n^{1/4})$$

Π is a shortest path system

$$\text{herdisc}(\Pi') = \tilde{\Theta}(n^{1/2})$$

Π' is some path system of polynomial size

- The new discrepancy lower bound improves the lower bound of DP problems

Open problem:

1. Close the gap for directed edge discrepancy $\tilde{\Omega}(n^{1/4})$ vs $O(n^{1/2})$ (Existential and explicit coloring)
2. Close the gap for DP-APSD additive error $\tilde{\Omega}(n^{1/4})$ vs $\tilde{O}(n^{1/2})$ (ε, δ -DP)

Questions?

Thank you!